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Particle and antiparticle sectors in DSR1 and κ-Minkowski space-time

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Abstract: In this paper we explore the problem of antiparticles in DSR1 and κ-Minkowski space-time following three different approaches inspired by the Lorentz invariant case: (a) the dispersion relation, (b) the Dirac equation in space-time and (c) the Dirac equation in momentum space. We find that it is possible to define a map S_{dsr} which gives the antiparticle sector from the negative frequency solutions of the wave equation. In κ-Poincaré, the corresponding map S_{kp} is the antipodal mapping, which is different from S_{dsr}. The difference is related to the composition law, which is crucial to define the multiparticle sector of the theory. This discussion permits to show that the energy of the antiparticle in DSR is the positive root of the dispersion relation, which is consistent with phenomenological approaches.

Keywords: Space-Time Symmetries, Models of Quantum Gravity.

Contents

1. Introduction

The idea that Lorentz symmetry could be modified at very high energies, owing to quantum gravity effects, has been intensively explored in the last years [1]. For example in string theory, the possibility that space-time could have a non-commutative structure [2] opens the door to a Lorentz non-invariant world [3]; in loop quantum gravity, on the other hand, neutrinos and photons might not satisfy a Lorentz invariant dispersion relation [4].

The fact that such violation of Lorentz invariance (LI) happens at certain distance (or energy) scale[1] and that this scale should be the same for any observer, led to the main idea of the so-called Doubly Special Relativity (DSR) principle, developed by G. Amelino-Camelia [5, 6] (DSR1) and by Magueijo and Smolin (DSR2) [7]. DSR has an explicit realization in momentum space, in the sense that the transformation laws for the energy and momentum of a particle in some reference frame are known in terms of a generalized boost [7, 8].

The realization of DSR in space-time is however a subtle problem. First, one notes that the algebra that characterizes DSR is the κ-Poincaré algebra (KP) introduced by Lukiersky, Ruegg, Nowicki and Tolstoi in ref. [9] and therefore, at the level of the algebras, different DSR theories correspond to different choices of basis in the KP context [10]. The space-time structure can be introduced in a natural way, by defining the dual algebra (Hopf algebra) of KP. The so-called κ-Minkowski (KM) space-time [11] constructed in such a way has a non-trivial algebra.

[1] The main candidate is the Planck scale.

But this construction of the space-time sector is not satisfactory at all from the DSR point of view. For example, if one considers a system of two particles, the four-momentum composition law inherited from the co-product structure is not invariant under the transformation laws of DSR [12].

On the other hand, we need to confront such proposals with experiments and predictions. DSR phenomenology has been studied [13] in order to test these new ideas from the kinematic point of view. That is, one starts with the deformed dispersion relation (for DSR1, see eq. (3.8)) plus a compatible composition law, and studies the new features that DSR implies.

In this approach, processes involving particles and antiparticles are very interesting because the assignment of the energy for antiparticles is not a trivial task. In the standard case, the Casimir $E^2 - \boldsymbol{p}^2$ has two roots which are symmetrical under the change $E \leftrightarrow -E$ and, therefore, there is no ambiguity in the interpretation of E as the energy of both the particle and the antiparticle. But in a DSR theory this interpretation is not so simple since in general there are two solutions for the energy, and they turn out to be non-symmetric [13].

In order to solve this problem one should construct the quantum field theory compatible with DSR. However, a first indication about the particle-antiparticle content of the theory can be traced back in the Dirac equation (DE) compatible with DSR principles.

A proposal for such operator has been given in ref. [14, 15] and before it, in the KP approach in ref. [16]. From both definitions of the Dirac operator one can extract information about the particle and antiparticle sectors and also contrast them in order to obtain a different perspective on the relation between KM space-time and DSR theories.

In this paper we will explore the particle and antiparticle sectors in DSR and KM space-time. In order to do that, we will first review three different approaches in the LI case: *a*) by studying the solutions for the energy in the dispersion relation; *b*) from the space-time solutions of the DE, and *c*) from the DE in momentum space. We will be able to follow approaches *a*) and *c*) to analyze the antiparticle sector in DSR. However, since we do not know how to write the DE compatible with DSR in space-time, we will study the KM deformed Dirac equation, which can also be written in momentum space. This will allow to compare the DSR and KM results.

The paper is organized as follows. In section 2 we will review the LI case, along the lines described in the previous paragraph. Section 3 is devoted to the analysis of the DE in DSR1 and KM space-time. Finally, in section 4 the discussion and conclusions are presented.

2. Lorentz invariant particle-antiparticle sectors

In this section we will review three approaches which permit to identify the particle and antiparticle sectors in the standard LI theory, namely: the roots of the energy in the dispersion relation, the DE in space-time, and the DE in momentum space.

These approaches are very well known and, of course, they are not enough to recognize and give a definite antiparticle interpretation, which becomes clear under the light of a quantum field theory (QFT), but we will use them as a guide to identify antiparticles in DSR theories since we do not have a proper QFT compatible with the DSR principle.

At the level of the Lorentz invariant theory, the DE in space-time is enough to introduce the concept of antiparticle; however, we will also consider the DE in momentum space in order to compare with DSR, for which the connection between space-time and momentum formulations is not known.

2.1 Dispersion relation

The first signal of the presence of antiparticles comes from the dispersion relation itself

$$E^2 - \boldsymbol{p}^2 = m^2\,, \tag{2.1}$$

which has two roots for the energy, $E_\pm = \pm\sqrt{m^2 + \boldsymbol{p}^2}$. The positive root E_+ corresponds to the particle energy, while E_-, being negative, appears to be classically meaningless as an energy of a free particle.

In quantum field theory, however, the negative energy solutions are connected with the existence of positive energy antiparticles, with energy $|E_-|$, which therefore gives a physical meaning to the solution coming from the negative sector. A useful picture to visualize the concept of antiparticle is that of the "Dirac sea". Here the vacuum consists of an infinitely deep sea of completely filled negative energy levels. An antiparticle is associated to a "hole" in the sea, that is, to a state in which all the negative energy levels but one are occupied. The hole corresponds to the absence of a negative energy E_-, and has therefore a larger energy than that of the vacuum, exactly in the quantity $-E_-$. Since we assign to the vacuum zero energy and momentum, the hole (antiparticle) associated to the absence of a negative energy state of momentum $\boldsymbol{p}$ and energy E_- satisfies

$$E_{\text{antip}} + E_- = 0\,, \qquad \boldsymbol{p}_{\text{antip}} + \boldsymbol{p} = 0\,. \tag{2.2}$$

The point that we would like to emphasize is that the sector of negative solutions (their modulus) in eq. (2.1) has a meaning and therefore, these solutions should not be discarded by hand. And the fact that $E_+ = |E_-|$ expresses a symmetry property, but it does not say that E_+ is the energy of the antiparticle, it only says that both energies coincide.

The role of both solutions becomes clearer from the Dirac equation, which we will review briefly in the next subsection.

2.2 Dirac equation in space-time

The Dirac equation in space-time is

$$(i\,\partial\!\!\!/ - m)\,\psi(x) = 0\,, \tag{2.3}$$

where$B\!\!\!/ = \gamma^\mu B_\mu$, γ^μ are the Dirac matrices and m is the mass of the particle.

The DE has two kind of linearly independent solutions

$$\psi^{(+)}(x) = e^{-ip_\mu x^\mu}\ u(p)\,,$$
$$\psi^{(-)}(x) = e^{ip_\mu x^\mu}\ v(p)\,, \tag{2.4}$$

with[2] $p_0 > 0$ and $u(p)$, $v(p)$ are bispinors which satisfy

$$(\not p - m)u(p) = 0\,,$$
$$(\not p + m)v(p) = 0\,. \tag{2.5}$$

In order to have non trivial solutions for $u(p)$ and $v(p)$ the condition (2.1) must be satisfied. Note that the condition $p_0 > 0$ implies that only positive roots must be considered.

Let us point out a fact that is trivial at this level, but that will be useful in the construction (and identification) of the antiparticle sector in DSR1. In eq. (2.3), $\psi^{(+)}(x)$ is called the positive-frequency solution, and describes a particle of energy p_0 and momentum $\boldsymbol{p}$, while $\psi^{(-)}(x)$ is named the negative-frequency solution, and corresponds to a negative energy $-p_0$ and a momentum $-\boldsymbol{p}$. The interpretation of this negative-frequency solution as an antiparticle follows the line of arguments of the previous subsection. Defining the map

$$S_{li}[(p^0, \boldsymbol{p})] = (-p^0, -\boldsymbol{p})\,, \tag{2.6}$$

we see that

1. S_{li} maps the positive-frequency solution $\psi^{(+)}(x)$ into the negative-frequency solution $\psi^{(-)}(x)$, and
2. S_{li} maps the energy-momentum of the negative frequency solution into the energy-momentum of the associated antiparticle state (see eq. (2.2)).

This trivial map satisfies the following properties:[3] *a*) $S_{li}[S_{li}[p]] = p$, *b*) $p + S_{li}[p] = 0$ and *c*) it leaves eq. (2.1) invariant.

Note that properties *a*) and *b*) lead to eq. (2.2) for the energy-momentum of the antiparticle, and properties *a*) and *c*) are necessary conditions for the two frequency solutions to be related by this mapping.

In the next subsection, we will discuss another approach to the DE, namely, the momentum space formulation. We will of course arrive to eq. (2.5), but without any explicit reference to space-time.

2.3 Dirac equation in the momentum space

We are interested in a DSR1 consistent approach; therefore, it is necessary the discussion of the problem in momentum space, where DSR is well understood and properly defined.

In the LI case, the DE can be obtained by performing a boost on, for instance, chiral bi-spinors $(u_L(0), u_R(0))$ defined in the rest frame with the condition $u_L(0) = u_R(0)$ [17].

[2]We are using the metric $\mathrm{diag}(1, -1, -1, -1)$.

[3]We will suppress the index μ in the four vectors in order to simplify the notation.

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Under a boost with rapidity ξ, the spinors transform according to

$$u_R(p) = \left[\cosh\left(\frac{\xi}{2}\right) + \boldsymbol{\sigma}\cdot\boldsymbol{n}\sinh\left(\frac{\xi}{2}\right)\right] u_R(0)\,,$$
$$u_L(p) = \left[\cosh\left(\frac{\xi}{2}\right) - \boldsymbol{\sigma}\cdot\boldsymbol{n}\sinh\left(\frac{\xi}{2}\right)\right] u_L(0), \tag{2.7}$$

where $\boldsymbol{n}$ is the direction of the momentum, σ^i $(i = 1,2,3)$ are the Pauli matrices, and ξ the rapidity, defined by

$$\cosh(\xi) = \frac{E}{m}, \qquad \sinh(\xi) = \frac{p}{m}\,.$$

The condition $u_L(0) = u_R(0)$ in eq. (2.7) becomes

$$(\not{p} - m)u(p) = 0\,, \tag{2.8}$$

that is, the DE for the particle sector, as discussed previously.

The antiparticle sector can be obtained from here by applying the map $S_{li}[p]$ discussed in the previous section, giving

$$(\not{p} + m)v(p) = 0\,, \tag{2.9}$$

where we have called $v(p) = Tu(S_{li}[p])$, and T is a matrix constructed from the γ matrices.

3. The particle-antiparticle problem in DSR and KM spacetime

In the present section we apply the preceding arguments to the DSR case. To do that, we first discuss a general approach to DSR in momentum space, and then we obtain the map S_{dsr} analogous to the map S_{li} of the Lorentz invariant case, giving its explicit form for the DSR1 model.

The map S_{dsr} will allow us to identify the antiparticle sector just from the dispersion relation of a DSR theory. Taking the DSR1 model as a specific example, we will then write the DSR1 DE in momentum space, as it was obtained in ref. [14], and from here we will be able to obtain the DE for the antiparticle.

The analysis of the DE in space-time compatible with the DSR principles is not an easy task, as mentioned in the Introduction. When compared in momentum space, the DE in KM space-time turns out to be the same as the DE in the DSR framework [14]. However, the antiparticle sector seems to be different in the two approaches, as we will see below.

3.1 DSR formulation

It is convenient for our purposes to understand DSR as a nonlinear realization of the Lorentz group. That is, we assume that there exists a function $F : \Re^4 \to \Re^4$ (with components F^μ) which acts on momentum space with coordinates $p^\mu = \{E, \boldsymbol{p}\}$, and maps it to the space χ_F with coordinates $\pi^\mu = \{\epsilon, \boldsymbol{\pi}\}$ as follows:[4]

$$\pi^\mu = \left(F^{-1}[p^\alpha]\right)^\mu\,,$$

in a way that the Lorentz group acts linearly on χ_F [7] and [18]–[20].

[4] The problem of identifying the original variables (p^μ) as the physical ones in an operational (measurable) way is still an open problem.

This function allows, for instance, to construct the finite DSR boosts ($\mathcal{B}$) from the standard Lorentz boosts Λ by means of the composition of functions,[5]

$$\mathcal{B} = F \circ \Lambda \circ F^{-1} .$$

The four-momentum composition law ($\hat{+}$) for two particles has to satisfy two physical conditions: i) it has to act linearly under the non-linear boosts and ii) it has to be invariant under the interchange of particle labels:

$$\mathcal{B}[p_a \,\hat{+}\, p_b] = \mathcal{B}[p_a] \,\hat{+}\, \mathcal{B}[p_b] \,, \tag{3.1}$$

$$\mathcal{B}[p_a \,\hat{+}\, p_b] = \mathcal{B}[p_b \,\hat{+}\, p_a] \,. \tag{3.2}$$

A natural definition that satisfies both conditions is given in terms of the F function:

$$p_a \,\hat{+}\, p_b = F\left[F^{-1}[p_a] + F^{-1}[p_b]\right] , \tag{3.3}$$

which gives a DSR consistent law [19].

In order to identify the antiparticle sector from the deformed DE, one needs to construct a map $S_{dsr}[p]$, demanding the same properties as in the LI case. However, one of these properties is rather non-trivial in this case because it involves the composition law, namely $p + S_{li}[p] = 0$, which has sense only if it is valid in any reference frame. The DSR compatible requirement will therefore be $p \hat{+} S_{dsr}[p] = 0$.

It is direct to prove that the map

$$S_{dsr}[p] = F\left[-F^{-1}[p]\right] , \tag{3.4}$$

satisfies all the requirements, that is: *a)* $S_{dsr}[S_{dsr}[p]] = p$, *b)* $p \hat{+} S_{dsr}[p] = 0$, and *c)* the Casimir element remains invariant.

For the DSR1 model, the explicit form of F and its inverse is

$$F[x, \boldsymbol{y}] = \begin{bmatrix} \frac{1}{\lambda} \ln\left(\lambda x + \sqrt{1 + \lambda^2 (x^2 - \boldsymbol{y}^2)}\right) \\ \boldsymbol{y}\left[\lambda x + \sqrt{1 + \lambda^2 (x^2 - \boldsymbol{y}^2)}\right] \end{bmatrix} , \tag{3.5}$$

$$F^{-1}[x, \boldsymbol{y}] = \begin{bmatrix} \frac{1}{\lambda}\left(\sinh(\lambda x) + \frac{\lambda^2}{2} \boldsymbol{y}^2 e^{\lambda x}\right) \\ \boldsymbol{y} e^{\lambda x} \end{bmatrix} . \tag{3.6}$$

And the map $S_{dsr}[p]$ (by components) becomes

$$\begin{aligned} S^0_{dsr}[p] &= -E + \frac{1}{\lambda} \ln\left(1 - \lambda^2 \boldsymbol{p}^2 e^{2\lambda E}\right) , \\ \boldsymbol{S}_{dsr}[p] &= \frac{-\boldsymbol{p}\ e^{2\lambda E}}{1 - \lambda^2 \boldsymbol{p}^2 e^{2\lambda E}} \,. \end{aligned} \tag{3.7}$$

It is interesting to note that this function S_{dsr} is the image of S_{li}, defined in the space χ_F, under the action of F. That is, in χ_F, the map connecting the particle-antiparticle sectors is the Lorentz invariant one: $S_{li}[\epsilon, \boldsymbol{\pi}] = (-\epsilon, -\boldsymbol{\pi})$. More precisely,

$$S_{dsr} = F \circ S_{li} \circ F^{-1} .$$

Now we are able to define the antiparticle sector in the DSR framework and contrast it with the results of the KM approach. The next sections are devoted to this discussion.

[5] We will suppress the index on F as we have done with p.

3.2 Dispersion relation approach

On a DSR1-type theory, the Casimir element can be written as

$$\frac{2}{\lambda^2}\cosh(\lambda E) - \boldsymbol{p}^2\, e^{\lambda E} = \frac{2}{\lambda^2}\cosh(\lambda m)\,, \tag{3.8}$$

where λ is the invariant scale. E and $\boldsymbol{p}$ are the energy and momentum of the particle. The parameter m is the rest mass of the particle.

The solutions of eq. (3.8) for the energy as a function of momentum are

$$E_\pm(\boldsymbol{p}) = \frac{1}{\lambda}\ln\left(\frac{\cosh(\lambda m) \pm \sqrt{\cosh^2(\lambda m) - (1-\lambda^2\boldsymbol{p}^2)}}{1-\lambda^2\boldsymbol{p}^2}\right), \tag{3.9}$$

and the LI case is recovered in the limit $\lambda \to 0$, where eq. (3.9) becomes $\pm\sqrt{\boldsymbol{p}^2+m^2}$, that is, both sectors of energies.

Note that in the rest frame ($|\boldsymbol{p}| = 0$), eq. (3.9) becomes

$$E_\pm(0) = \pm m\,, \tag{3.10}$$

without any corrections due to DSR. The result does not depend on λ and therefore, the interpretation of E_+ as the energy of the particle and $|E_-|$ as the energy of the antiparticle should remain unchanged in this frame.

After a DSR1 boost, the energies of the particle and its antiparticle can be read from eq. (3.9). The solution E_- remains negative in any reference frame and this could suggest to associate this solution with an antiparticle of energy $-|E_-|$. However, this guess assumes implicitly that the energy and momentum of the antiparticle are obtained from the negative-frequency solution $(E_-(\boldsymbol{p}), -\boldsymbol{p})$ by applying the map S_{li}. But this approach is too naive: we should be consistent and use S_{dsr} instead. Then the energy of the antiparticle becomes

$$\begin{aligned} S^0_{dsr}[E_-] &= -E_- + \frac{1}{\lambda}\ln\left[e^{-\lambda E_-}\cosh(\lambda m) - 1\right] \\ &\sim -E_- - \lambda(E_-^2 - m^2) - \mathcal{O}(\lambda^2)\,. \end{aligned} \tag{3.11}$$

From the previous equation we see that $S^0_{dsr}[E_-]$ is positive and, since the mapping S_{dsr} leaves the Casimir invariant, it results that $S^0_{dsr}[E_-(\boldsymbol{p})] = E_+(\boldsymbol{S}_{dsr}[\boldsymbol{p}])$. On the other hand, since $E_-(\boldsymbol{p})$ does not depend on the sign of $\boldsymbol{p}$, $S^0_{dsr}[E_-(\boldsymbol{p})] = S^0_{dsr}[E_-(-\boldsymbol{p})] = E_+(\boldsymbol{S}_{dsr}[-\boldsymbol{p}])$.

Therefore, the particle state $(E_+, \boldsymbol{p})$ is associated with the antiparticle state $(W, \boldsymbol{q}) = (S^0_{dsr}[E_-(\boldsymbol{p})], \boldsymbol{S}_{dsr}[-\boldsymbol{p}])$ where $W = E_+(\boldsymbol{q}) > 0$. Both the particle and the antiparticle have the same relation between energy and momentum, i.e. in both cases the dispersion relation ($C(E_+, \boldsymbol{p}) = 0$ and $C(W, \boldsymbol{q}) = 0$) is defined by the same Casimir invariant $C(p)$ and the energy dependence on the spatial momentum is given by the positive solution of eq. (3.9). This is not a trivial result and it was crucial to use the properly defined S_{dsr} instead of the usual S_{li}.

3.3 DSR1 Dirac equation

The DSR1 version of the DE can be obtained in the same way described in section 2.3, but now considering the DSR1 boost [14]

$$\cosh(\xi) = \frac{e^{\lambda E} - \cosh(\lambda m)}{\sinh(\lambda m)}. \tag{3.12}$$

By using also the Casimir (3.8), the DSR-DE becomes

$$(\not{D} - \ \sinh(\lambda m))\, u(p) = 0\,, \tag{3.13}$$

where

$$D_0^\lambda = e^{\lambda E} - \cosh(\lambda m)\,, \tag{3.14}$$

$$D_j^\lambda = \lambda\, e^{\lambda E}\ p_j\,. \tag{3.15}$$

Note that from the point of view of our approach to DSR (section 3.1) it is possible to obtain the same equation (3.13) just by mapping the DE from the space χ_F to the space where the Lorentz group acts nonlinearly.[6] In fact eq. (3.13) can be rewritten as

$$\left[\gamma^\mu F_\mu^{-1}[p] - \frac{\sinh(\lambda m)}{\lambda}\right] u(p) = 0\,. \tag{3.16}$$

The antiparticle sector can be obtained as follows. From a pure DSR1 point of view we must consider the map $S_{dsr}[p]$ defined in eq. (3.7), under which the DE becomes

$$\left[\gamma^\mu F_\mu^{-1}\left[S_{dsr}[p]\right] - \frac{\sinh(\lambda m)}{\lambda}\right] v(p) = 0\,. \tag{3.17}$$

But taking into account that $S_{dsr} = F \circ S_{li} \circ F^{-1}$, one has

$$F^{-1}\left[S_{dsr}[p]\right] = S_{li} \circ F^{-1}[p] = -F^{-1}[p]$$

and then the antiparticle spinor $v(p)$ satisfies the equation

$$\left(\gamma^0 \left[e^{\lambda E} - \cosh(\lambda m)\right] + \gamma^i p_i\ e^{\lambda E} + \sinh(\lambda m)\right) v(p) = 0\,. \tag{3.18}$$

Again, as in the case of particles, the DE for antiparticles is the image of eq. (2.5) in the χ_F space under the action of F.

In the next section we will analyze the problem in KM space-time and we will see that a different map is needed in order to render the antiparticle sector.

[6] One should take into account that the parameter m in the DSR1 theory is such that $\frac{\sinh(\lambda m)}{\lambda}$ is the generalization of the Lorentz invariant mass.

3.4 The κ-Minkowski space-time approach

In KM space-time, things turn out to be more complicated because we have non-commutativity in the sense[7]

$$[x_i, x_0] = i\lambda\ x_i\,,$$

with $\{i = 1, 2, 3\}$, and x_0 the time coordinate.

In order to find solutions of the deformed DE, we will follow ref. [22], using the $\star$-product formulation. The algebra of functions is given by

$$(f \star g)\,(x) = f(x)g(x) - \frac{i\lambda}{2}\Big(x^\ell \partial_\ell f(x)\partial_0 g(x) - x^\ell \partial_\ell g(x)\partial_0 f(x)\Big)\ldots, \tag{3.19}$$

with λ the DSR invariant scale.

The derivatives in the non-commutative space-time $\hat{\partial}_\mu$ are mapped to standard derivatives in the commutative space-time as follows:

$$\hat{\partial}_0 \longrightarrow \partial_0^* = \partial_0\,, \tag{3.20}$$

$$\hat{\partial}_j \longrightarrow \partial_j^* = \frac{e^{-i\lambda\partial_0} - 1}{-i\lambda\partial_0}\partial_j\,. \tag{3.21}$$

The Dirac operator (DO) is constructed in the non-commutative space-time by demanding that it transforms as a vector, and the requirement that it has the standard DO limit when $\lambda \to 0$. The image of the DO in the commutative space turns out to be

$$D_0^* = \frac{1}{\lambda}\sin(\lambda\partial_0) - \frac{i\nabla^2}{\lambda\partial_0^2}\left(\cos(\lambda\partial_0) - 1\right), \tag{3.22}$$

$$D_j^* = \frac{e^{i\lambda\partial_0} - 1}{i\lambda\partial_0}\partial_j\,, \tag{3.23}$$

where $\nabla^2 = \partial_j\ \partial_j$.

In order to find solutions of the DE, let us define the four-momentum-like operator Π_μ, on the commuting space-time

$$\Pi_j = \frac{1 - e^{-i\lambda\partial_0}}{-\lambda\partial_0}\ \partial_j\,,$$

$$\Pi_0 = i\partial_0\,. \tag{3.24}$$

One can verify that the class of functions

$$\phi^{(+)}(x) = e^{-i\left[E\,t - i\left(\frac{\lambda E}{1-e^{-\lambda E}}\right)\boldsymbol{p}\cdot\boldsymbol{x}\right]}, \tag{3.25}$$

satisfies

$$\Pi_\mu\ \phi^{(+)}(x) \;=\; p_\mu\ \phi^{(+)}(x)\,, \tag{3.26}$$

$$\phi^{(+)}(x) \overset{\lambda\to 0}{\longrightarrow} e^{-i\ p_\mu x^\mu}\,, \tag{3.27}$$

with $p^\mu = (E, \boldsymbol{p})$. The last equation suggests to interpret this function as the one corresponding to the particle sector solutions.

[7]Note that the sign of λ is reversed if we compare with the original work of Wess [22]. The reason for doing that is two fold: first, the sign of λ is not fixed in the κ-Minkowski deformation and second, in order to match with the results given, for instance in ref. [21].

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It is also straightforward to prove that this class of solutions also satisfies

$$D_0^* \; \phi^{(+)}(x) \; = \; -\frac{i}{\lambda} \left[e^{\lambda E} - \cosh(\lambda m) \right] \; \phi^{(+)}(x) \,, \tag{3.28}$$

$$D_i^* \; \phi^{(+)}(x) \; = \; -i \; e^{\lambda E} p_i \; \phi^{(+)}(x) \,, \tag{3.29}$$

where we have used the κ-Poincaré dispersion relation which is the same as eq. (3.8) [22].

Therefore, the function

$$\psi(x) = \phi^{(+)}(x) \; u(p) \,, \tag{3.30}$$

with $\phi^{(+)}(x)$ given in eq. (3.25), is a solution of the DE

$$\left(i \, \not{D}^* - \lambda^{-1} \sinh(\lambda m) \right) \psi(x) = 0 \,, \tag{3.31}$$

provided that $u(p)$ is a solution of

$$\left\{ \left[e^{\lambda E} - \cosh(\lambda m) \right] \gamma^0 + \lambda \; e^{\lambda E} \; p_i \gamma^i - \sinh(\lambda m) \right\} u(p) = 0 \,. \tag{3.32}$$

It is clear that the condition to have a non-trivial solution for u is the dispersion relation given by eq. (3.8). Finally, we conclude that the energy of the particle sector corresponds to the positive root of the deformed dispersion relation, as we expected.

As was noted in ref. [14], we also see that the momentum space equation (3.32), derived from KM space-time, coincides with the DSR1 Dirac equation (3.13).

In order to find the other sector, the antiparticle one, we must look for the solutions with the limit $e^{ip_\mu x^\mu}$ for $\lambda \to 0$. In doing this we will proceed as in the previous cases using the S mapping as a natural tool.

In defining S_{kp} for KP we first need the composition law for four-momentum. This can be derived from the coproduct structure of the KP model and has the peculiarity to be non-symmetric:

$$E_a \dot{+} E_b = E_a + E_b \qquad \boldsymbol{p}_a \dot{+} \boldsymbol{p}_b = \boldsymbol{p}_a + e^{\lambda E_a} \boldsymbol{p}_b \,. \tag{3.33}$$

In principle this gives the possibility to have two different definitions of S_{kp}, i.e.

$$p_a \dot{+} S_{kp}[p_a] = 0 \qquad S_{kp}[p_a] \dot{+} p_a = 0 \,.$$

It is remarkable that both definitions give the same result

$$S_{kp}^0 \; [p] \; = \; -E \,, \tag{3.34}$$

$$\boldsymbol{S}_{kp} \; [p] \; = \; -\boldsymbol{p} \; e^{\lambda E} \,. \tag{3.35}$$

This map is well known in the context of Hopf algebras and it corresponds to the antipodal mapping.

As it was shown in the LI case and used in the DSR1 model, in this case we will demand again that the antiparticle four-momentum $(W, \boldsymbol{q})$ be related to the four-momentum of the negative frequency solution $p \equiv (E_-, \boldsymbol{p})$ by $W = S_{kp}^0 \; [p]$ and $\boldsymbol{q} = \boldsymbol{S}_{kp} \; [p]$.

As a consequence of the general properties defining the S mapping one has that, although the relation between the energy-momentum of the negative frequency solution and the energy-momentum of the antiparticle differs from the DSR1 case, once more the energy of the antiparticle depends on the momentum in the same way as the energy of the particle.

The property $S[S[p]] = p$ allows to write p in terms of q and obtain the solution of the DE for the antiparticle as follows. The function

$$\phi^{(-)}(x) = e^{i\left[W\,t - \left(\frac{\lambda W}{1-e^{\lambda W}}\right)\boldsymbol{q}\cdot\boldsymbol{x}\right]}, \tag{3.36}$$

satisfies

$$\Pi_\mu\ \phi^{(-)}(x) \;=\; -q_\mu\ \phi^{(-)}(x)\,, \tag{3.37}$$

$$\phi^{(-)}(x) \xrightarrow{\lambda\to 0} e^{i\ q_\mu x^\mu}\,. \tag{3.38}$$

It is direct to prove that the function

$$\psi(x) = \phi^{(-)}(x)\ v(p)\,, \tag{3.39}$$

with $\phi^{(-)}(x)$ given by eq. (3.36), is a solution of the DE

$$\left(i\,\not{D}^{*} - \lambda^{-1}\sinh(\lambda m)\right)\psi(x) = 0\,, \tag{3.40}$$

provided that $v(p)$ is a solution of

$$\left\{\left[e^{-\lambda w} - \cosh(\lambda m)\right]\gamma^0 - \lambda\ p_i\gamma^i - \sinh(\lambda m)\right\}v(q) = 0\,. \tag{3.41}$$

The DE for the antiparticle spinor is different from the one obtained in the DSR1 model. This is due to the different composition laws, which induce different S mappings.

4. Conclusions and discussion

In this paper we have studied the antiparticle sector in the context of the DSR1 proposal and the κ-Minkowski space-time, following three different approaches inspired by the Lorentz invariant case: *a*) the dispersion relation, *b*) the Dirac equation in space-time and *c*) the Dirac equation in momentum space.

For the DSR proposal, whose consistent realization in space-time is not known, just *a*) and *c*) are well defined. On the other hand, there is evidence that, at least in the one particle sector, the KM approach could be compatible with the DSR principles and therefore, the test *b*) could give another insight on such relation.

In section 2, in a LI analysis, it is possible to realize that the sector of particles and antiparticles are connected by a map (S_{li}), which is trivial in a LI theory, with the properties pointed out in section 2.2. The interesting point is that one of such properties requires the composition law: a rule for summing momentum that must be covariant. When this law is the standard sum of four vectors, the covariance is guaranteed by the linear nature of Lorentz transformations.

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From a pure DSR point of view, we showed that it is possible to define a map S_{dsr} which gives the antiparticle sector. Such map turns out to be the image of S_{li} by the function F in the approach of [7, 19, 20] to DSR.

Then, following a), we showed that for DSR1 one should take the positive root of the dispersion relation as the energy of the antiparticle. The negative solution has not a physical meaning; however, when it is mapped by S_{dsr} it renders the antiparticle energy. But because the spatial momentum is also mapped in a non-trivial way, the energy of the antiparticle depends on its spatial momentum in the same way as the energy of the particle depends on its momentum, and both are positive.[8] This result is consistent with some phenomenological approaches [13, 23].

In the approach c), from the DE in momentum space (as was obtained in ref. [14], for instance), the antiparticle sector is straightforwardly obtained through the map S_{dsr}. We found that the DE for the antiparticle is the image under F of the standard DE (for antiparticles) in the space χ_F.

From the space-time point of view (approach b)), it is possible to find a solution of the deformed DE in KM space-time which, in momentum space, is the same as the DE compatible with DSR1.

However, here we can not use the map S_{dsr} to obtain the antiparticle sector. In fact, if we look for a function S with the properties discussed in section 2.2, there is a crucial difference, that is, the rule for summing momenta is not $\hat{+}$, the DSR one; here instead, such rule is given by the coproduct structure. This fixes the map S, (which turns out to be the antipodal map, in the Hopf algebra language) and because of that the antiparticle sector in KP is different from that of DSR1.

This difference is expressed in the Dirac equation. The deformed DE satisfied by a DSR1 antiparticle is different from the deformed DE satisfied by a KM particle. The dispersion relations are the same, but clearly their physical content is rather different. This physical content, at the end, is codified in the composition law which is crucial for any physical process.

Let us comment that our result can be viewed as another argument showing the independence between DSR and KM theories (and κ-Poincaré). However, it clearly shows that the ideas coming from the last one might give some light in the construction of a DSR in space-time.

As a final remark, the identification of the S map between particle and antiparticle sectors suggests a generalization,

$$\Psi(x) = \int \frac{d^3\boldsymbol{F}[p]}{(2\pi)^3} \frac{1}{2F_0^{-1}[p]} \left[a_p u(p) e^{-ip.x} + b_p^\dagger v(p) e^{-iS[p].x}\right],$$

of the relativistic Dirac quantum field of a Lorentz invariant theory as an starting point for a quantum field formulation of a DSR theory.

[8] In fact in the case of a spinless system, the identification of the S map at the level of the dispersion relation is all one needs to establish the relation between the particle and antiparticle sectors.

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References

[1] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos, D.V. Nanopoulos and S. Sarkar, *Tests of quantum gravity from observations of gamma-ray bursts*, *Nature* **393** (1998) 763 [astro-ph/9712103];
R. Gambini and J. Pullin, *Nonstandard optics from quantum spacetime*, *Phys. Rev.* **D 59** (1999) 124021 [gr-qc/9809038];
T. Jacobson, S. Liberati and D. Mattingly, *TeV astrophysics constraints on Planck scale Lorentz violation*, *Phys. Rev.* **D 66** (2002) 081302 [hep-ph/0112207];
V.A. Kostelecký and C.D. Lane, *Constraints on Lorentz violation from clock-comparison experiments*, *Phys. Rev.* **D 60** (1999) 116010 [hep-ph/9908504];
R. Jackiw and V.A. Kostelecký, *Radiatively induced Lorentz and cpt violation in electrodynamics*, *Phys. Rev. Lett.* **82** (1999) 3572 [hep-ph/9901358];
D. Colladay and V.A. Kostelecký, *Lorentz-violating extension of the standard model*, *Phys. Rev.* **D 58** (1998) 116002 [hep-ph/9809521]; *Cross-sections and Lorentz violation*, *Phys. Lett.* **B 511** (2001) 209 [hep-ph/0104300];
V.A. Kostelecký and R. Lehnert, *Stability, causality and Lorentz and CPT violation*, *Phys. Rev.* **D 63** (2001) 065008 [hep-th/0012060];
R. Bluhm and V.A. Kostelecký, *Lorentz and CPT tests with spin-polarized solids*, *Phys. Rev. Lett.* **84** (2000) 1381 [hep-ph/9912542];
J. Alfaro and G. Palma, *Loop quantum gravity corrections and cosmic rays decays*, *Phys. Rev.* **D 65** (2002) 103516 [hep-th/0111176];
R. Aloisio, P. Blasi, P.L. Ghia and A.F. Grillo, *Probing the structure of space-time with cosmic rays*, *Phys. Rev.* **D 62** (2000) 053010 [astro-ph/0001258];
R. Aloisio, P. Blasi, A. Galante, P.L. Ghia and A.F. Grillo, *Space time fluctuations and ultrahigh-energy cosmic ray interactions*, *Ann. Phys. (NY)* **19** (2003) 127 [astro-ph/0205271]; *A fluctuating energy-momentum may produce an unstable world*, *Ann. Phys. (NY)* **20** (2003) 369;
J.M. Carmona, J.L. Cortes, J. Gamboa and F. Mendez, *Noncommutativity in field space and Lorentz invariance violation*, *Phys. Lett.* **B 565** (2003) 222 [hep-th/0207158]; *Quantum theory of noncommutative fields*, *J. High Energy Phys.* **03** (2003) 058 [hep-th/0301248].

[2] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162]; *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[3] S.M. Carroll, J.A. Harvey, V.A. Kostelecky, C.D. Lane and T. Okamoto, *Noncommutative field theory and Lorentz violation*, *Phys. Rev. Lett.* **87** (2001) 141601 [hep-th/0105082].

[4] J. Alfaro, H.A. Morales-Tecotl and L.F. Urrutia, *Quantum gravity corrections to neutrino propagation*, *Phys. Rev. Lett.* **84** (2000) 2318 [gr-qc/9909079]; *Loop quantum gravity and light propagation*, *Phys. Rev.* **D 65** (2002) 103509 [hep-th/0108061].

[5] G. Amelino-Camelia, *Relativity in space-times with short distance structure governed by an observer independent (Planckian) lenght scale*, *Int. J. Mod. Phys.* **D 11** (2002) 35; *Status of relativity with observer-independent length and velocity scales*, gr-qc/0106004.

[6] G. Amelino-Camelia, *Doubly special relativity: first results and key open problems*, *Int. J. Mod. Phys.* **D 11** (2002) 1643.

[7] J. Magueijo and L. Smolin, *Lorentz invariance with an invariant energy scale*, *Phys. Rev. Lett.* **88** (2002) 190403 [hep-th/0112090]; *Generalized Lorentz invariance with an invariant energy scale*, *Phys. Rev.* **D 67** (2003) 044017 [gr-qc/0207085].

[8] N.R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, *Deformed boost transformations that saturate at the planck scale*, *Phys. Lett.* **B 522** (2001) 133 [hep-th/0107039].

[9] J. Lukierski, H. Ruegg, A. Nowicki and V.N. Tolstoi, *Q deformation of Poincaré algebra*, *Phys. Lett.* **B 264** (1991) 331; *Real forms of complex quantum anti-de Sitter algebra* $U_q(\mathrm{Sp}(4;C))$ *and their contraction schemes*, *Phys. Lett.* **B 271** (1991) 321 [hep-th/9108018]; J. Lukierski, H. Ruegg and W. Ruhl, *From kappa Poincaré algebra to kappa Lorentz quasigroup: a deformation of relativistic symmetry*, *Phys. Lett.* **B 313** (1993) 357.

[10] J. Kowalski-Glikman and S. Nowak, *Doubly special relativity theories as different bases of kappa-Poincaré algebra*, *Phys. Lett.* **B 539** (2002) 126 [hep-th/0203040].

[11] S. Majid and H. Ruegg, *Bicrossproduct structure of kappa Poincaré group and noncommutative geometry*, *Phys. Lett.* **B 334** (1994) 348 [hep-th/9405107];
S. Zakrzewski, *Quantum Poincaré group related to the K-Poincaré algebra*, *J. Phys.* **A 27** (1994) 2075.

[12] G. Amelino-Camelia, *The three perspectives on the quantum-gravity problem and their implications for the fate of Lorentz symmetry*, gr-qc/0309054.

[13] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici and A. Procaccini, *Phenomenology of doubly special relativity*, gr-qc/0312124;
J. Kowalski-Glikman, *Doubly special relativity and quantum gravity phenomenology*, hep-th/0312140.

[14] A. Agostini, G. Amelino-Camelia and M. Arzano, *Dirac spinors for doubly special relativity*, *Class. and Quant. Grav.* **21** (2004) 2179 [gr-qc/0207003].

[15] D.V. Ahluwalia-Khalilova, *Fermions, bosons, and locality in special relativity with two invariant scales*, gr-qc/0207004; *Special relativity with two invariant scales: motivation, fermions, bosons, locality, and critique*, *Int. J. Mod. Phys.* **D 13** (2004) 335 [gr-qc/0402023].

[16] A. Nowicki, E. Sorace and M. Tarlini, *The quantum deformed Dirac equation from the kappa Poincaré algebra*, *Phys. Lett.* **B 302** (1993) 419 [hep-th/9212065];
L.C. Biedenharn, B. Muller and M. Tarlini, *The Dirac-Coulomb problem for the kappa Poincaré quantum group*, *Phys. Lett.* **B 318** (1993) 613 [hep-th/9310173];
H. Ruegg and V.N. Tolstoi, *Representation theory of quantized Poincaré algebra: tensor operators and their applications to one particle systems*, *Lett. Math. Phys.* **32** (1994) 85 [hep-th/9406146];
J. Lukierski, H. Ruegg and W.J. Zakrzewski, *Classical quantum mechanics of free kappa relativistic systems*, *Ann. Phys. (NY)* **243** (1995) 90 [hep-th/9312153].

[17] L.H. Ryder, *Quantum field theory*, Cambridge University Press, 1996, pp. 37–40.

[18] J. Lukierski and A. Nowicki, *Doubly special relativity versus κ-deformation of relativistic kinematics*, *Int. J. Mod. Phys.* **A 18** (2003) 7 [hep-th/0203065].

[19] S. Judes and M. Visser, *Conservation laws in 'doubly special relativity'*, *Phys. Rev.* **D 68** (2003) 045001 [gr-qc/0205067].

[20] R. Schutzhold and W.G. Unruh, *Problems of doubly special relativity with variable speed of light*, gr-qc/0308049.

[21] J. Kowalski-Glikman and S. Nowak, *Noncommutative space-time of doubly special relativity theories*, *Int. J. Mod. Phys.* **D 12** (2003) 299 [hep-th/0204245].

[22] M. Dimitrijevic et al., *Deformed field theory on kappa-spacetime*, *Eur. Phys. J.* **C 31** (2003) 129 [hep-th/0307149];
M. Dimitrijevic, F. Meyer, L. Moller and J. Wess, *Gauge theories on the kappa-Minkowski spacetime*, hep-th/0310116.

[23] D. Heyman, F. Hinteleitner and S. Major, *On reaction thresholds in doubly special relativity*, gr-qc/0312089.